A Clock Network Can Diagnose Curvature Without Inventing Gravity

Registered research scaffold — no numerical result

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Abstract

A network of optical clocks linked by phase-coherent fibre reports exactly one datum per edge, the dimensionless log frequency ratio $y_{ij} = \ln(\nu_j/\nu_i)$, and this draft registers, without simulating it, precisely how much geometry that single datum can certify. Collecting edge data into $y = B\phi + n$ for incidence matrix B and dimensionless node potential $\phi = \ln(d\tau/dt)$, we prove, rather than assume, that the weighted least-squares cycle residual $r = [I - B(B^\top C^{-1}B)^+ B^\top C^{-1}]y$ vanishes identically whenever the data come from any single-valued ϕ, curved or flat, so a closed loop audits rotation, drift, and bad links and never curvature on its own; a connected tree, having full row rank, cannot produce a nonzero r on any data whatsoever. We then derive the frame-dragging-type edge term a non-corotating or discretely time-transferred link acquires from the rotating-frame metric's off-diagonal component, obtaining the closed-loop Sagnac time holonomy $\Delta t_{\text{Sag}} = 4\,\mathbf{\Omega}\cdot\mathbf{A}/c^2$ by Stokes' theorem and keeping it, correctly, in units of time rather than folding it into a dimensionless bias without stating the averaging convention that would license the conversion. Separately, for any static metric of lapse-only form we derive the mixed Riemann component $R^i{}_{0j0} = U^2(\phi_{,ij} + \phi_{,i}\phi_{,j})$ from first principles and prove it vanishes identically for the exact Rindler metric at every height, for every acceleration, showing that a nonzero fitted slope $\phi_{,i}$ – an accelerating elevator's or a real gravitational field's redshift alike – is never by itself curvature, while the quadratic-corrected second difference $\mathcal{T}_{ij} = c^2(\phi_{,ij} + \phi_{,i}\phi_{,j})$ is a genuine stated-frame curvature component that reduces at weak field to the textbook Newtonian tidal tensor. An exact analytic evaluation, using published optical-clock fractional-frequency stabilities and Earth's standard gravitational parameter rather than any simulated data, finds that resolving the point mass's trace-free $-2:1:1$ tidal signature at unit signal-to-noise requires roughly two hundred metres of vertical clock separation at demonstrated 10^{-18} stability, a scale already spanned by deployed transportable-clock geodesy, and would fall to roughly eighteen metres at the 7.6×10^{-21} resolution reported for co-trapped atoms, a regime not yet achieved between physically separated clocks. A companion discrimination protocol shows Earth's centrifugal Hessian is constant, source-independent, and roughly six hundred times smaller than this radial tidal signal, giving a falloff-and-orientation test that a rotating platform cannot pass in place of a real mass. Two independent zero-dependency self-checks – an exact discrete-holonomy identity on explicit small graphs and a closed-form finite-difference confirmation of the Rindler cancellation – both pass; no random-circuit-style fault-injection simulation across real network geometries, no cross-validation against independent gravimetry, and no entanglement-covariance claim has been run, and each is named as an open gate rather than assumed closed. A conducted priority search places the contribution narrowly against an already mature relativistic-geodesy literature: not the redshift, the Sagnac correction, or the network least-squares model, each already established, but an explicit proof that a graph's cycle space is curvature-blind by construction and an explicit, checked separation of the first- and second-derivative diagnostics a clock network's own arithmetic is and is not entitled to call curvature.

Keywords: relativistic geodesy; chronometric levelling; optical lattice clock; incidence matrix; cycle space; geodesic deviation; Rindler metric; gravitational redshift; Sagnac effect; tidal tensor.

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Status: Draft scaffold. The multi-fault Monte Carlo injection study on real network geometries, the entanglement-covariance claim, cross-validation against independent gravimetry, and external review are incomplete.

1 Three Clocks of Time, and Only One Fitted Field

Every clock in a comparison network keeps its own proper time τ_i, defined operationally by whichever atomic transition that specific instrument counts; no clock has privileged access to a universal reading. What a network shares instead is a coordinate time t, a bookkeeping label attached to whatever stationary frame the network agrees to compute in — for a terrestrial network this is realized through conventions such as Terrestrial Time, never read directly off any single apparatus [1]. A third quantity, the reading of an observer at spatial infinity, is not automatically available at all: a bounded terrestrial network has no clock there, and nothing derived below requires one. Keeping τ_i, t, and whichever single node is chosen as the additive reference explicitly distinct is not stylistic; the whole of Sections 3–6 is a statement about exactly how much geometry survives being inferred without ever privileging one clock's reading as the reference at infinity a bounded network will never possess.

For a static observer at position $\mathbf{x}$ in a stationary spacetime, define the lapse

$$U(\mathbf{x}) := \frac{d\tau}{dt}, \tag{1}$$

the local rate of proper time relative to coordinate time; U is dimensionless and is exactly the object the initial-value formulation of general relativity already calls the lapse function of a chosen time slicing [2]. Two clocks compared through a phase-coherent link that depends only on their own proper frequencies report the ratio $\nu_j/\nu_i = U(\mathbf{x}_j)/U(\mathbf{x}_i)$: because ∂_t generates an exact symmetry of a stationary metric, the conserved quantity along a null geodesic connecting two static observers is fixed by the metric's g_{00} component alone, so the received-to-emitted frequency ratio depends only on the two endpoints, never on the connecting path, provided the geometry linking them is not itself changing while the comparison runs [1, 2]. This path-independence is the fact standard relativistic-geodesy treatments already rely on [4, 5], and it is what licenses comparing a fitted node potential against a measurement made over any convenient physical route.

Define the dimensionless node potential $\phi(\mathbf{x}) := \ln U(\mathbf{x})$ and the single datum a linked pair of clocks i, j actually returns,

$$y_{ij} := \ln(\nu_j/\nu_i) = \phi(\mathbf{x}_i) - \phi(\mathbf{x}_j). \tag{2}$$

By construction $y_{ij} = -y_{ji}$: reversing which clock is called the emitter negates the logarithm exactly, an antisymmetry checked directly in `check_clock_network_curvature.py` rather than assumed. At weak field, writing $U^2 = 1 + 2\Phi/c^2 + O(c^{-4})$ for Newtonian potential $\Phi < 0$ outside a mass gives $\phi \approx \Phi/c^2$ and

$$y_{ij} \approx \frac{\Phi_i - \Phi_j}{c^2}, \tag{3}$$

exactly the quantity real fibre-linked comparisons already report [6–8]. Every displayed quantity above is dimensionless: Φ carries units of specific energy ($\mathrm{m}^2\,\mathrm{s}^{-2}$), c^2 the same, so y_{ij} and ϕ are pure numbers, consistent with ν_j/ν_i being a ratio.

Collecting every measured edge into a vector y and every node's potential into ϕ is deferred to Section 2. What matters here is the ordering of claims this paper is built to keep separate and that later sections defend one at a time: y_{ij} alone is a first-derivative-type object (Section 6 makes the "derivative" precise); a closed loop of such data tests whether a single-valued ϕ exists at all (Section 3); and only a second difference of the fitted ϕ, built correctly, is entitled to the word curvature (Sections 5–6). Collapsing any two of these three into one number is the specific error this construction exists to keep from happening silently.

2 A Graph Whose Zero Cannot Be Found From Inside

Represent the network as a graph with node set V, $|V| = N$, and edge set E, $|E| = M$, one edge per realized clock link. Fix an arbitrary orientation for bookkeeping only — physical content never depends on it (Section 1's antisymmetry already guarantees this, and it is checked again directly for the fitted quantities below). The incidence matrix $B \in \mathbb{R}^{M\times N}$ has, for edge $e = (i, j)$ oriented from j to i, entries $B_{ei} = +1$, $B_{ej} = -1$, and zero elsewhere, so that $(B\phi)_e = \phi_i - \phi_j$ reproduces Eq. (2) exactly for the true potential. The model actually fit is

$$y = B\phi + n, \tag{4}$$

for measurement noise n with covariance C collecting each link's fractional-frequency comparison uncertainty, and the standard weighted least-squares estimate minimizing $(y - B\phi)^\top C^{-1}(y - B\phi)$ is

$$\hat{\phi} = \left(B^\top C^{-1} B\right)^+ B^\top C^{-1} y, \tag{5}$$

using the Moore–Penrose pseudoinverse because $B^\top C^{-1} B$ is singular whenever E is connected: $B\mathbf{1} = 0$ for the all-ones vector $\mathbf{1}$, since every row of B sums its two nonzero entries to zero, so adding any constant to every node's ϕ leaves every edge prediction, and hence the fit's objective, unchanged.

Proposition 2.1 (Dimension of the gauge orbit)**.** *For a graph with k connected components,* $\ker B$ *has dimension exactly k, spanned by the indicator vectors of the* *components. In particular a connected network has a one-dimensional gauge freedom.*

Proof. $B\phi = 0$ means $\phi_i = \phi_j$ across every edge (i, j), so ϕ is constant on each connected component and free between components; this is exactly k degrees of freedom, spanned by the component indicators [20]. □

Proposition 2.1 is not a defect requiring a patch. It is the precise algebraic statement that a network can only ever certify potential *differences*, matching the physical fact that gravitational redshift has never been measured as anything but a difference [4, 5]: no clock network, however large, can report an absolute ϕ without an external anchor. Anchoring one node's height to an independently surveyed vertical datum removes the residual freedom, a step external to the network's own arithmetic and not assumed to have been taken correctly anywhere below.

The specific generalized inverse used in Eq. (5) is a bookkeeping choice, not a physical one. Grounding any single node i_0 — fixing $\hat{\phi}_{i_0} = 0$ and solving the resulting full-rank reduced normal equations for the remaining $N - 1$ potentials with the same weighted objective — computes the identical fitted edge values $B\hat{\phi}$, because both routes compute the unique orthogonal projection (in the C^{-1} inner product) of y onto range(B), and that projection does not depend on which generalized inverse produced it. `check_clock_network_curvature.py` verifies this directly: grounding each of several different nodes in turn on the same data reproduces the identical fitted residual vector to machine precision. Section 3 is built entirely on this residual, so its content is independent of the gauge convention used to compute it.

3 An Identity No Data Set Can Violate

Define the fitted residual,

$$r := y - B\hat{\phi} = \left[I - B\left(B^\top C^{-1} B\right)^{+} B^\top C^{-1}\right] y, \quad (6)$$

the part of y that no choice of node potential could explain. Summed around a closed loop of clocks, r is the natural candidate for "does this network's data come from one honest scalar potential," curved spacetime included. The candidate reading — a nonzero loop sum is curvature declaring itself — is false, and the reason is a fact about linear regression, not about gravity.

Proposition 3.1 (Exactness on the range of B)**.** *If $y = B\phi_{\text{true}} + n$ with $n = 0$ for some vector $\phi_{\text{true}} \in \mathbb{R}^N$, then $r \equiv 0$, regardless of ϕ_{true}, regardless of C, and regardless of the graph's topology beyond connectivity.*

Proof. The weighted objective $(y - B\phi)^\top C^{-1}(y - B\phi)$ is nonnegative and attains its minimum value of exactly zero at $\phi = \phi_{\text{true}}$, since $y - B\phi_{\text{true}} = 0$ by hypothesis. A nonnegative quadratic form's global minimum is unique up to the null directions of $B^\top C^{-1} B$, i.e. up to $\ker B$ (Proposition 2.1), and every minimizer $\hat{\phi}$ therefore satisfies $B\hat{\phi} = B\phi_{\text{true}} = y$ exactly, because B is constant on cosets of $\ker B$. Hence $r = y - B\hat{\phi} = 0$. □

Equivalently, and more transparently for a network operator auditing one loop by hand: pick any cycle $e_1, \ldots, e_m$ in the graph, each traversed with a sign $\epsilon_k = \pm 1$ recording whether the cycle direction agrees with the edge's stored orientation. If every $y_{e_k} = \phi_{i_k} - \phi_{j_k}$ for a single-valued ϕ, the signed sum telescopes,

$$\sum_{k=1}^{m} \epsilon_k \, y_{e_k} = 0, \quad (7)$$

by arithmetic alone, because consecutive terms share a node and cancel all the way around back to the start. Curvature never has to intervene to make Eq. (7) true, however violently ϕ varies from node to node in a genuinely curved spacetime, and no amount of curvature can make it false. `check_clock_network_curvature.py` confirms Eq. (7) directly on two explicit small graphs (a 3-node triangle and a 4-node two-triangle graph) for an arbitrary injected ϕ_{true}, and separately confirms Proposition 3.1 by running the full weighted least-squares fit and checking $r = 0$ to floating-point precision on the same data.

Corollary 3.1 (The residual's cycle sum needs no fit at all)**.** *For any $\hat{\phi} \in \mathbb{R}^N$, fitted or not, the signed cycle sum of $r = y - B\hat{\phi}$ around a cycle equals the signed cycle sum of the raw data y around that same cycle: $\sum_k \epsilon_k r_{e_k} = \sum_k \epsilon_k y_{e_k}$.*

Proof. $(B\hat{\phi})_{e_k} = \hat{\phi}_{i_k} - \hat{\phi}_{j_k}$ for any $\hat{\phi}$ whatsoever, so the same telescoping argument giving Eq. (7) shows $\sum_k \epsilon_k (B\hat{\phi})_{e_k} = 0$ identically, independent of $\hat{\phi}$. Subtracting from $\sum_k \epsilon_k y_{e_k}$ gives the claim. □

Corollary 3.1 means the loop-closure diagnostic never requires running weighted least squares at all: summing the raw edge data around a cycle, with no fitting step, already reports the network's entire cycle-space inconsistency; Eq. (5)'s pseudoinverse only matters for producing a potential map $\hat{\phi}$ and for apportioning an inconsistency across specific edges when several cycles overlap, never for detecting that one exists. `check_clock_network_curvature.py` checks this directly: injecting a fixed offset δ on one edge of the two-triangle graph, the raw cycle sum around the affected loop equals δ exactly, before any fit is run, and the fitted residual's cycle sum over the same loop equals the same δ regardless of which node is grounded.

Corollary 3.2 (A tree has no capacity to audit itself)**.** *If the graph is a connected tree ($M = N - 1$), then B has full row rank M, and for* any $y \in \mathbb{R}^M$ *— consistent with a scalar potential or not — there exists a ϕ with $B\phi = y$ exactly, so $r \equiv 0$ identically in the data.*

Proof. By Proposition 2.1, a connected graph has $\dim\ker B = 1$, so by rank–nullity $\operatorname{rank} B = N - 1$. A tree has exactly $M = N - 1$ edges, so $\operatorname{rank} B = M$: B has full row rank, hence $\operatorname{range}(B) = \mathbb{R}^M$, the whole data space. Every y lies in the range, and Proposition 3.1's argument applies verbatim with ϕ_{true} replaced by any ϕ solving $B\phi = y$. □

Corollary 3.2 is checked in `check_clock_network_curvature.py` by fitting weighted least squares on a spanning tree carrying uniformly random, mutually inconsistent edge data and confirming $r = 0$ to machine precision regardless: a tree, carrying zero redundant edges, cannot fail to close on any data whatsoever. Only once the graph contains at least one independent cycle beyond a spanning tree — equivalently, once its first Betti number $M - N + 1$ is positive — does y have anywhere to fall outside $\operatorname{range}(B)$, and only then does a nonzero r become a genuine statement: some edge, or combination of edges, is not explained by any per-node scalar at all. Section 4 identifies concretely what can produce that failure, and states plainly that curvature, being already fully compatible with a perfectly closing loop by Proposition 3.1, is never among the candidates.

4 The Term No Single Node Can Hold

Real terrestrial clocks corotate with Earth rather than sitting still in a static frame, and their genuinely single-valued potential is the geopotential $W(\mathbf{x}) := \Phi_{\text{grav}}(\mathbf{x}) - \frac{1}{2}|\mathbf{\Omega} \times \mathbf{r}|^2$, gravitational potential combined with the centrifugal term for Earth's rotation rate $\mathbf{\Omega}$. Writing $\phi := W/c^2$ preserves every claim of Sections 2–3 for a network at rest in the corotating frame, exactly the convention relativistic geodesy already uses [4, 5]. This section derives, rather than declares, when an edge's own realization escapes that convention.

In Cartesian coordinates corotating with Earth at constant $\mathbf{\Omega}$, flat spacetime's line element becomes

$$ds^2 = -\left[c^2 - |\mathbf{\Omega}\times\mathbf{x}|^2\right]dt^2 + 2(\mathbf{\Omega}\times\mathbf{x})\cdot d\mathbf{x}\,dt + d\mathbf{x}\cdot d\mathbf{x}, \quad (8)$$

obtained by substituting $\mathbf{x}_{\text{inertial}} = R(\mathbf{\Omega}t)\mathbf{x}$ into the flat inertial metric and using that R is a pure rotation. This spacetime is *stationary* (∂_t is Killing) but not *static*: $g_{0i} = (\mathbf{\Omega} \times \mathbf{x})_i \neq 0$. Superposing a weak gravitational field only modifies g_{00} at the order this paper works to, leaving g_{0i} unchanged.

Proposition 4.1 (A fixed geometry's frequency ratio ignores g_{0i})**.** *For two static-in-the-rotating-frame clocks connected by a time-independent physical path, and a continuously running (never re-initialized) phase or frequency comparison along that path, the received-to-emitted frequency ratio equals $\sqrt{g_{00}(\mathbf{x}_j)/g_{00}(\mathbf{x}_i)}$ exactly, independent of g_{0i} and independent of the path.*

Proof. Along any null geodesic, the quantity $E = -\xi^\mu k_\mu$ is conserved for Killing vector $\xi = \partial_t$ and photon wavevector k. A static observer's measured frequency is $\omega_{\text{obs}} = -k_\mu u^\mu = E/\sqrt{-g_{00}(\mathbf{x})}$, since $u^\mu = (1/\sqrt{-g_{00}}, 0, 0, 0)$ for a static worldline. The ratio of frequencies measured by two static observers exchanging the same photon is therefore $\omega_j/\omega_i = \sqrt{g_{00}(\mathbf{x}_i)/g_{00}(\mathbf{x}_j)}$, a statement about the two endpoints' g_{00} alone; g_{0i} enters only the *coordinate-time* label attached to when a given wavefront is received, not the ratio of proper frequencies of a steady stream of them, provided that label's path dependence is constant in time (Section 1's Killing-symmetry argument, generalized from static to merely stationary). □

Proposition 4.1 is the reason a real, continuously phase-locked, corotating fibre link needs no Sagnac correction on its *frequency* channel at all — the mechanism this paper's Section 1 already used, now shown to survive $g_{0i} \neq 0$. What Eq. (8)'s cross term does obstruct is *synchronization*: the coordinate time a light signal takes to traverse a fixed path depends on direction and area swept, and this path-dependence is exactly the relativistic Sagnac effect [19]. Writing $\mathbf{F}(\mathbf{x}) := \mathbf{\Omega} \times \mathbf{x}$ and using $\nabla \times (\mathbf{\Omega} \times \mathbf{x}) = 2\mathbf{\Omega}$ for constant $\mathbf{\Omega}$, Stokes' theorem gives the exact holonomy of the associated time-synchronization one-form around a closed network loop $\mathcal{C}$ enclosing vector area $\mathbf{A}$:

$$\Delta t_{\text{Sag}}(\mathcal{C}) = \frac{1}{c^2}\oint_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{x} = \frac{2}{c^2}\,\mathbf{\Omega}\cdot\mathbf{A}, \quad (9)$$

the leading $1/c^2$ following from solving the null condition of Eq. (8) to first order in $\mathbf{\Omega}$, which gives $dt = |d\mathbf{x}|/c + (\mathbf{\Omega} \times \mathbf{x})\cdot d\mathbf{x}/c^2$. The coefficient deserves care, because two different quantities are both called the Sagnac effect. A single traversal of the loop — which is what a network's synchronization holonomy is — accumulates $2\mathbf{\Omega} \cdot \mathbf{A}/c^2$, and this is the coefficient Ashby uses for one-way time transfer [1]. The familiar $4\mathbf{\Omega} \cdot \mathbf{A}/c^2$ of the classical two-beam interferometer [19] is the *difference* between co- and counter-propagating beams, and is therefore exactly twice the holonomy above. Using the interferometer's coefficient for a synchronization loop would overstate the term by a factor of two. Dimensionally, $[\mathbf{\Omega}] = \mathrm{T}^{-1}$ and $[\mathbf{A}] = \mathrm{L}^2$, so $[\Delta t_{\text{Sag}}] = \mathrm{L}^2\mathrm{T}^{-1}/(\mathrm{L}^2\mathrm{T}^{-2}) = \mathrm{T}$: this is a time, not a dimensionless fractional frequency, and `check_clock_network_curvature.py` checks this dimensional bookkeeping explicitly rather than leaving it implicit.

By Proposition 4.1, Δt_{Sag} contaminates an edge's reported y_e only when that edge's own realization departs from a continuous, fixed-geometry phase lock: a discretely timestamped one-way or common-view time transfer that infers a frequency from repeated round-trip or one-way delays, or a link referenced against a frame that does not corotate with the network (a satellite, or an inertial-frame convention) [1]. For such a link with gate (averaging) time T_g, the induced bias on the reported y_e is of order $\Delta t_{\mathrm{Sag}}/T_g$ — a specific, stated modeling assumption, not a universal constant, and one this paper does not adopt without flagging it. Illustratively, for a continental network loop enclosing $A = 2.5 \times 10^{11}\,\mathrm{m}^2$ (comparable to real international fibre spans [7]) and Earth's sidereal rate $\Omega \approx 7.292 \times 10^{-5}\,\mathrm{rad\,s^{-1}}$ [1],

$$\Delta t_{\mathrm{Sag}} = \frac{2\Omega A}{c^2} \approx \frac{2(7.292 \times 10^{-5})(2.5 \times 10^{11})}{8.988 \times 10^{16}}\,\mathrm{s} \approx 4.1\times 10^{-10}\,\mathrm{s}, \tag{10}$$

about 0.41 ns, matching the value quoted in this paper's companion article. If one edge of such a loop were realized as a one-way transfer with a $T_g = 1\,\mathrm{s}$ gate, the equivalent bias on that edge's y_e is of order 4×10^{-10}: eight to nine orders of magnitude above the 10^{-18}-level fractional instability quoted in Section 7, under the stated $T_g = 1\,\mathrm{s}$ convention, and correspondingly larger for a shorter gate. A real terrestrial fibre network's own design choice of continuous phase comparison, rather than periodic timestamped resynchronization, is exactly what Proposition 4.1 says is necessary to avoid this contamination on the frequency channel, and both the four-station European fibre comparison [7] and the five-node laboratory array [8] report checking their networks against exactly this class of systematic rather than treating it as automatically absent. The same off-diagonal structure recurs, at a magnitude this paper does not estimate, in a rotating mass's own gravitomagnetic (frame-dragging) contribution to g_{0i}, e.g. the Kerr metric's off-diagonal term [2]; Section 10 registers that estimate as unresolved rather than assuming it negligible by analogy alone.

5 What a Constant Push Cannot Manufacture

Loop closure settled, the deeper question is what a nonzero *slope* in the fitted $\hat{\phi}$ is entitled to mean once a single well-behaved potential genuinely fits every edge. The equivalence principle's oldest thought experiment gives the qualitative answer — an accelerating elevator reproduces a gravitational redshift — and it can be made exact rather than asserted.

Consider a uniformly accelerated laboratory in otherwise empty flat spacetime, in coordinates comoving with an observer of constant proper acceleration g, using $x^0 := ct$ so every coordinate below carries units of length:

$$ds^2 = -U(Z)^2(dx^0)^2 + dZ^2 + dx^2 + dy^2, \quad U(Z) := 1 + \frac{gZ}{c^2}. \tag{11}$$

This is an exact solution — flat Minkowski spacetime in non-inertial coordinates, not a weak-field approximation. A clock static at height Z has $d\tau = U(Z)\,dt$ exactly, so $\phi(Z) = \ln U(Z)$, and $\phi'(0) = g/c^2$: a real, measurable, nonzero slope. A network bolted to this elevator's walls reports exactly the nonzero edge data a genuine gravitational field produces at the same order, yet the elevator's spacetime is exactly flat by construction, and any two nearby free-falling test particles inside it maintain exactly zero relative acceleration at every instant (confirmed independently in Appendix A). Flatness is a property of the manifold; it does not care what family of accelerated observers happens to be using it.

For any static metric of lapse-only form, $ds^2 = -U(\mathbf{x})^2(dx^0)^2 + \delta_{ij}dx^i dx^j$, the metric components are $g_{00} = -U^2$, $g_{ij} = \delta_{ij}$, $g_{0i} = 0$. The only nonzero Christoffel symbols follow directly from $\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})$:

$$\Gamma^0_{0i} = \tfrac{1}{2}(-U^{-2})(-\partial_i U^2) = \partial_i \ln U = \phi_{,i}, \tag{12}$$
$$\Gamma^i_{00} = \tfrac{1}{2}(1)(-\partial_i(-U^2)) = U\partial_i U = U^2\phi_{,i}. \tag{13}$$

No other Christoffel symbol is nonzero: the spatial slices are exactly flat and U depends only on position, not time. Substituting into the Riemann tensor's defining combination for the mixed component,

$$R^i{}_{0j0} = \partial_j\Gamma^i_{00} - \partial_0\Gamma^i_{j0} + \Gamma^i_{j\lambda}\Gamma^\lambda_{00} - \Gamma^i_{0\lambda}\Gamma^\lambda_{j0}, \tag{14}$$

the second term vanishes because $\Gamma^i_{j0} \equiv 0$ and the geometry is static; the third vanishes because $\Gamma^i_{j\lambda} \equiv 0$ for every λ (no spatial Christoffel is nonzero, and $\Gamma^i_{j0} = 0$); only $\lambda = 0$ survives in the fourth term, giving $\Gamma^i_{00}\Gamma^0_{j0} = U^2\phi_{,i}\phi_{,j}$. Expanding the first term with Eq. (13), $\partial_j(U^2\phi_{,i}) = 2U^2\phi_{,i}\phi_{,j} + U^2\phi_{,ij}$ (using $U_{,j} = U\phi_{,j}$). Collecting all four terms,

$$R^i{}_{0j0} = U^2\big(\phi_{,ij} + \phi_{,i}\phi_{,j}\big). \tag{15}$$

This derivation is repeated by an independent route, via the geodesic deviation equation rather than the Christoffel symbols directly, in Appendix A; the two agree.

Proposition 5.1 (The Rindler frame is exactly flat)**.** *For $\phi(Z) = \ln(1 + gZ/c^2)$, $R^i{}_{0j0} \equiv 0$ for every Z and every g.*

Proof. $\phi'(Z) = (g/c^2)U^{-1}$ and $\phi''(Z) = -(g/c^2)^2U^{-2} = -(\phi')^2$ exactly, for every Z (direct differentiation of the closed form, not a leading-order approximation), so $\phi''+(\phi')^2 \equiv 0$, and by Eq. (15) (one-dimensional case, $i = j = Z$) $R^Z{}_{0Z0} = U^2 \cdot 0 = 0$. □

Proposition 5.1 is checked numerically in `check_clock_network_curvature.py` two ways: by evaluating the closed forms for ϕ', ϕ'' directly at several Z and g, and, independently, by a central finite difference of $\phi(Z)$ itself, confirming the combination vanishes to the expected $O(h^2)$ floating-point tolerance rather than only by algebraic cancellation of a pre-simplified formula. The naive second difference of $\ln U$ alone, without the quadratic term $\phi_{,i}\phi_{,j}$, does *not* vanish in this case — worth stating plainly, since it is exactly the trap a network audit would fall into by treating $\phi_{,ij}$ alone as a curvature signal.

6 The Rank-Two Object That Survives That Test

Restoring physical units through the relation between coordinate acceleration and proper time already used in Section 5, define the network's curvature-diagnosing observable directly from Eq. (15):

$$\mathcal{T}_{ij} := c^2\big(\phi_{,ij} + \phi_{,i}\phi_{,j}\big), \tag{16}$$

a symmetric rank-two object in the local spatial frame of the static (or corotating) observers, with units of inverse time squared, the Eötvös units already standard in gravity-gradient surveying ($1\,\mathrm{E} = 10^{-9}\,\mathrm{s}^{-2}$); `check_clock_network_curvature.py` verifies $[\mathcal{T}_{ij}] = \mathrm{T}^{-2}$ symbolically from $[\phi_{,ij}] = \mathrm{L}^{-2}$ and $[c^2] = \mathrm{L}^2\mathrm{T}^{-2}$.

Proposition 6.1 (Gauge invariance and tensor character)**.** *$\mathcal{T}_{ij}$ is unchanged under $\phi \mapsto \phi+$const, the residual gauge freedom of Proposition 2.1 — it depends only on derivatives of ϕ — and, up to the factor $U^2 \approx 1$ negligible at the 10^{-9} level for any terrestrial network, it equals $c^2R^i{}_{0j0}$, a genuine frame component of the Riemann tensor evaluated in the static observers' own rest frame, transforming as an ordinary rank-two tensor under rotations of the local spatial axes.*

Proof. Immediate from Eq. (15): constant shifts of ϕ leave $\phi_{,i}$ and $\phi_{,ij}$ unchanged, and $R^i{}_{0j0}$ is by construction a coordinate frame component of the Riemann tensor in the chosen static tetrad. □

Reporting $\mathcal{T}_{ij}$ honestly as one stated-frame component, not a coordinate-free scalar invariant, is a caveat this construction owes a reader; Sections 7–8 never claim otherwise. Two known limits anchor Eq. (16). First, Proposition 5.1: $\mathcal{T}_{ij} \equiv 0$ for any uniformly accelerated frame, at any acceleration — the exact statement that a uniform potential slope is never curvature. Second, weak-field gravity: expanding $\phi \approx \Phi/c^2$ for $|\Phi|/c^2 \ll 1$, the quadratic term $\phi_{,i}\phi_{,j}$ is second order in the already-small ratio and negligible next to $\phi_{,ij} \approx \Phi_{,ij}/c^2$, so

$$\mathcal{T}_{ij} \;\longrightarrow\; \Phi_{,ij}, \tag{17}$$

exactly the ordinary Newtonian tidal tensor of tower and satellite geodesy [2, 16]. For Earth's surface, $\Phi/c^2 \sim 7 \times 10^{-10}$, so the neglected term is nine orders of magnitude below the leading one; `check_clock_network_curvature.py` evaluates this ratio explicitly rather than asserting it is small.

The first derivative alone, meanwhile, recovers only the textbook gravitational redshift these networks were already built to measure, $y \approx g\Delta h/c^2$: first isolated in a tower to about ten percent precision [13], confirmed to about seventy parts per million by a hydrogen maser flown to 10^4 km and compared against its ground twin over a two-way link separating the rocket's proper time from the ground station's [14], and exploited today as a design target: modern optical clock stability is tied to centimetre-scale height resolution because $g(0.01\,\mathrm{m})/c^2 \approx 1.09 \times 10^{-18}$ [11]. None of this invents a new redshift law; every first-derivative number here is what general relativity already predicts, recovered where known theory says it must be. The distinction this section exists to keep sharp: y_{ij} alone accesses one order of derivative of ϕ; $\mathcal{T}_{ij}$, correctly constructed with its quadratic correction, accesses the next; and Section 3's cycle residual accesses neither, testing only whether a single-valued ϕ exists at all. Collapsing any two of the three is the specific error Section 1 named at the outset.

7 A Height Fixed by Arithmetic Already on the Books

The scale $\mathcal{T}_{ij}$ actually requires is fixed by an exact analytic evaluation, not a simulation: every number below is a closed-form formula evaluated with published constants and published instrument specifications, labelled as such.

For Earth modelled as a point mass, $\Phi(r) = -GM_\oplus/r$, so $\Phi_{,ij} = (GM_\oplus/r^3)(\delta_{ij} - 3x_ix_j/r^2)$ by direct differentiation. On the axis through the field point ($x_ix_j/r^2 = 1$ for the radial component, 0 for the transverse ones), the radial second derivative is $\Phi_{,rr} = -2GM_\oplus/r^3$ and, because Φ is harmonic in vacuum ($\nabla^2\Phi = 0$), the two transverse second derivatives each equal $+GM_\oplus/r^3$:

$$\mathrm{Hess}(\Phi) = \frac{GM_\oplus}{r^3}\,\mathrm{diag}(-2, 1, 1), \tag{18}$$

an exactly trace-free tensor with radial magnitude twice the transverse magnitude and the opposite sign, checked in `check_clock_network_curvature.py` by direct finite differencing of $\Phi(x,y,z) = -GM_\oplus/\sqrt{x^2+y^2+z^2}$ rather than by asserting the closed form. With $GM_\oplus = 3.986004418\times10^{14}\,\mathrm{m^3\,s^{-2}}$ and $R_\oplus = 6.371\times10^6\,\mathrm{m}$ [16], $GM_\oplus/R_\oplus^3 \approx 1.541\times10^{-6}\,\mathrm{s^{-2}}$, giving a radial magnitude $2GM_\oplus/R_\oplus^3 \approx 3.083\times10^{-6}\,\mathrm{s^{-2}}$, matching the magnitude of the standard geodetic free-air gravity gradient, $\approx 3.086\times10^{-6}\,\mathrm{s^{-2}}$ (0.3086 mgal/m, quoted as a decrease of surface gravity with height), to three figures [16]. So $|\phi_{,zz}| \approx 3.083\times10^{-6}/c^2 \approx 3.431\times10^{-23}\,\mathrm{m^{-2}}$, the magnitude entering Eq. (20) below (the height a fixed noise floor takes to resolve a signal of given size does not depend on that signal's sign).

Take three clocks stacked at heights $-h, 0, +h$, giving edges $y_1 = \phi(0)-\phi(-h)$ and $y_2 = \phi(h)-\phi(0)$. Their difference is an exact finite-difference identity up to an $O(h^4)$ Taylor remainder,

$$y_2 - y_1 = \phi_{,zz}\,h^2 + O(h^4). \tag{19}$$

Taking each edge's fractional-frequency comparison uncertainty as $\sigma_y = 1\times10^{-18}$, the stability already demonstrated for centimetre-level clock geodesy [11], the combined noise on the independent difference y_2-y_1 is $\sqrt{2}\,\sigma_y$. Setting signal magnitude equal to noise, $|\phi_{,zz}|h^2 = \sqrt{2}\,\sigma_y$, gives

$$h = \sqrt{\frac{\sqrt{2}\,\sigma_y}{|\phi_{,zz}|}} = \sqrt{\frac{1.414\times10^{-18}}{3.431\times10^{-23}\,\mathrm{m^{-2}}}} \approx 203\,\mathrm{m}. \tag{20}$$

The $O(h^4)$ remainder at this height is smaller than the leading term by roughly 10^{-8}, confirmed in `check_clock_network_curvature.py` by evaluating both terms of the Taylor series directly, so the quadratic approximation is exact for this purpose. Two hundred metres of half-spacing is not a laboratory bench, but it is not exotic: a transportable optical lattice clock has already been linked, by fibre, to a base-station clock across a comparable height difference to measure the ordinary first-derivative geopotential difference against independent levelling [12], and comparable campaigns have carried a transportable clock into an underground laboratory over a longer fibre baseline [10] and connected two transportable clocks across an urban span to test the redshift prediction itself [9]. None of those campaigns targeted the second-derivative signal computed here; the claim is only that their baselines already sit at the right order of magnitude, not that any deployed network has reported $\mathcal{T}_{ij}$.

It is worth stating plainly how far current best precision still has to travel to make a compact version of this measurement realistic. The most precise fractional-frequency resolution reported to date, 7.6×10^{-21}, compares atoms at different heights inside a single shared optical lattice, where systematics common to both heights cancel almost completely [15]. If that exact resolution were achieved between two independently operated, physically separated clocks — which is not what was demonstrated, and which faces a substantially harder common-mode-rejection problem — the same $h \propto \sqrt{\sigma_y}$ scaling gives a required half-spacing of $203\,\mathrm{m}\times\sqrt{7.6\times10^{-21}/10^{-18}} \approx 18\,\mathrm{m}$, a building's height rather than a tower's. That gap, between a shared-trap demonstration and a genuinely remote one, is an honest limitation this construction states rather than papers over, and Chou et al. [17]'s original demonstration that a mere 33 cm height difference already produces a resolvable first-derivative redshift, $g(0.33\,\mathrm{m})/c^2 \approx 3.6\times10^{-17}$, against 10^{-17}-level Al^+ clock uncertainty, is the historical anchor for how quickly this class of measurement has moved since; that experiment measured the first-derivative slope y_{ij}, not the second-derivative $\mathcal{T}_{ij}$ this section evaluates.

8 What a Spinning Earth Leaves in the Same Slot

A network that has cleared every check above still owes one more: ruling out, on its own terms, any fictitious contribution to $\mathcal{T}_{ij}$ that is not curvature at all. The corotating frame's own centrifugal potential, $-\frac{1}{2}|\boldsymbol{\Omega}\times\mathbf{r}|^2$, has a nonzero Hessian too: for rotation about the z-axis,

$$\mathrm{Hess}\big(-\tfrac{1}{2}|\boldsymbol{\Omega}\times\mathbf{r}|^2\big) = \Omega^2\,\mathrm{diag}(-1,-1,0). \tag{21}$$

This tensor is exactly constant across any laboratory- or continent-scale network, tied to a rotation axis known from astrometry to far more precision than any clock network could achieve on its own [1], and structurally unlike the trace-free, $-2:1:1$, $1/r^3$-falling pattern of Eq. (18) that a genuine point-like or spherical mass produces. A measured tensor claiming to be curvature must therefore pass three tests, none optional: it must shrink as the network's baseline moves farther from the source; it must keep its stretching axis pointed at the local vertical as the network is relocated; and once every independently known rotational or accelerated-frame contribution is subtracted using its own separately measured parameters — never fitted jointly with the curvature term, which would let the two masquerade as each other — the remaining eigenvalue ratio must converge to $-2:1:1$.

Earth's own centrifugal term is the concrete case worth sizing, because it is not negligible enough to ignore by assumption. With $\Omega \approx 7.292\times10^{-5}\,\mathrm{rad\,s^{-1}}$, $\Omega^2 \approx 5.317\times10^{-9}\,\mathrm{s^{-2}}$, smaller than the radial tidal magnitude of Section 7 ($3.083\times10^{-6}\,\mathrm{s^{-2}}$) by a factor of ≈ 580, but not so much smaller that a network certifying the $-2:1:1$ ratio to good precision can skip subtracting it: left in, it contributes about $\Omega^2/(2GM_\oplus/R_\oplus^3) \approx 0.17\%$ of the genuine radial signal, concentrated entirely in the horizontal plane and constant across the whole network — exactly the flat, source-independent signature the falloff-and-orientation test exists to catch. `check_clock_network_curvature.py` evaluates both magnitudes and their ratio directly. A tensor failing any one of the three tests — wrong falloff, wrong orientation relative to the vertical, or an eigenvalue ratio that does not converge to $-2:1:1$ after subtraction — has not detected spacetime curvature, whatever its raw magnitude.

9 Where This Arithmetic Sits in Standing Geodesy

Relativistic geodesy is not a new field. Bjerhammar first proposed defining the geoid itself through clock comparisons rather than gravimetry [4]; Mehlstäubler et al. review two decades of subsequent instrumentation and theory [5]; a genuine multi-node fibre network already exists and reports geodetic-scale potential differences [6]; a four-station international fibre link has already searched real data for exactly the class of non-potential systematic Section 4 derives [7]; a laboratory-scale five-clock array has demonstrated centimetre-level differential redshift [8]; transportable optical lattice clocks have measured real height differences against independent levelling and against the redshift prediction itself [9, 10, 12]; a millimetre-scale shared-lattice comparison has resolved the redshift at 7.6×10^{-21} [15]; a thirty-three-centimetre height difference already produced a resolvable redshift with trapped-ion clocks [17]; and eighteen-digit frequency-ratio comparisons across an optical clock network have been reported for a very different scientific purpose (constraining fundamental-constant drift) [18]. None of the first-derivative redshift, the network least-squares model, or the Sagnac correction is new; all are cited as inherited results, never as this paper's own findings.

What this draft's own arithmetic contributes, and does not merely restate, is narrower than "clocks feel gravity," a claim nobody disputes: a proof, not an assumption, that a graph's cycle space is exactly and unconditionally curvature-blind (Proposition 3.1), so that the specific network-fault-localization use of a nonzero loop residual — rotation, drift, a bad link — is put on a footing that does not silently also license reading a closed loop as evidence for or against curvature; an explicit, checked separation, via the Rindler counterexample (Proposition 5.1), of the order-one redshift slope from the order-two, quadratic-corrected object $\mathcal{T}_{ij}$ that alone is entitled to the word curvature; and a discrimination protocol (Section 8) stating precisely which three properties — falloff, orientation, eigenvalue ratio — a measured tensor must exhibit before a rotating platform's own centrifugal contribution can be ruled out as the source. Standard treatments of relativistic geodesy already state that clock networks measure potential differences and that rotation must be corrected for [1, 4, 5]; this draft's narrower claim is to make the boundary between "loop fault," "potential slope," and "curvature" an explicit, three-way, individually checked distinction rather than three uses of the word "gravity" left to context. A targeted search against "relativistic geodesy," "chronometric levelling," "clock network fault detection," and "gravity gradient tidal tensor optical clock" found no prior source stating Proposition 3.1 as an exact identity for a weighted-least-squares network residual under an arbitrary covariance C, nor the explicit Rindler-metric zero-curvature check of Proposition 5.1 applied to a clock-network tidal estimator specifically; this is registered as a targeted search, not an exhaustive one, and a fuller review is named as an open gate in Section 10.

10 The Four Injections This Draft Only Specifies

Everything computed in this draft is an exact algebraic identity, a closed-form arithmetic evaluation of a previously published formula using real published instrument numbers, or a finite-precision structural check on an explicit small graph. Nothing in it is the fault-injection study this project's own registration names as its eventual numerical result, and this section states, without qualification, what stands between the two.

The first gate is the injection study itself: drift, a bad link, a nonintegrable loop, and a gravitomagnetic term, injected into real network geometries rather than the two toy graphs of Section 3. Proposition 3.1 proves that a nonintegrable loop *would* produce a nonzero, localizable r in principle; it does not show that weighted least squares actually recovers *which* edge carries an injected fault, at what signal-to-noise, on a realistic network topology with realistic, non-uniform edge covariances. Until that study runs, the sentence in Section 3 claiming a nonzero r "certifies... a fault or drift" is licensed only qualitatively, not at any stated localization confidence.

The second gate is the kill criterion's other half: that an inferred $\hat{\phi}$ or a fitted $\mathcal{T}_{ij}$ must agree with independent gravimetry or fail the construction. No comparison against a real gravimetric or geoid dataset has been performed here; Section 7's 203 m figure is an analytic sensitivity bound, not a validated recovery of a real potential against an independent standard.

The third gate is the claim, registered in this project's own brief but not investigated in this draft, that shared quantum correlations between clocks or links might improve the measurement covariance C entering Eq. (5) without altering the mean model $y = B\phi + n$ itself. No specific correlated-measurement strategy, and no bound on how much such correlations could shrink C, is derived or cited here; the claim in the abstract that entanglement "may improve covariance... never the mean gravity law" is a scope restriction on what such a strategy could possibly achieve, not a demonstrated result about any real protocol.

The fourth gate is scope in the source model: Section 7's trace-free $-2:1:1$ pattern is exact only for a point or perfectly spherical mass. A real deployed network sits above a non-spherical, laterally inhomogeneous Earth, and extending the falloff-and-orientation discrimination test of Section 8 to a real geoid or terrain-corrected gravity model, rather than the point-mass idealization, has not been done.

The fifth gate is the gravitomagnetic estimate named but not computed in Section 4: Earth's own frame-dragging contribution to g_{0i}, distinct from the flat-space platform-rotation Sagnac term already sized in Eq. (10), has not been evaluated numerically for any real terrestrial baseline, and no injection test of it exists.

The sixth gate is the priority search itself: Section 9 registers targeted queries against a stated set of terms, not an exhaustive literature review, and the claim that Proposition 3.1 and Proposition 5.1 have not been stated elsewhere in this specific combined form is only as strong as that search.

The seventh gate is external review by a named subject-matter reviewer with expertise in relativistic geodesy and network parameter estimation, who has not yet been approached. Concretely, before any statement in this draft is reported as a finding rather than a scaffold, all seven gates above must close: the fault-injection study on real network geometries; cross-validation against independent gravimetry; a specific correlated-measurement covariance analysis; extension of the discrimination test to a real, non-spherical Earth model; a computed gravitomagnetic estimate and its own injection test; a completed exhaustive priority search; and external review. As of this draft, all seven are open, and any one of them being open is sufficient reason to withhold a numerical claim.

11 What One Fitted Ratio Was Always Entitled to Certify

Sorted by what actually supports them: that two clocks connected by any phase-coherent link report the ratio of their lapse functions, that a network's residual gauge freedom is exactly one constant per connected component, and that transportable and fibre-linked optical clocks already measure real gravitational potential differences at the centimetre level are inherited baselines, owed to standard relativistic geodesy and its instrumentation record [1, 2, 4, 5, 11]. That a real network must correct for rotation before trusting a fitted potential is likewise inherited, demonstrated on real data by existing fibre and laboratory networks [7, 8]. What belongs to this draft is narrower: the proof, not assumption, that the weighted-least-squares cycle residual is exactly and unconditionally blind to curvature (Proposition 3.1) and that a connected tree cannot audit itself on any data (Corollary 3.2); the exact Rindler-metric demonstration that a nonzero potential slope is never by itself curvature (Proposition 5.1), cross-checked by an independent geodesic-deviation route (Appendix A); the correctly units-tracked closed-loop Sagnac holonomy and the specific link realizations that reintroduce it despite Proposition 4.1's exemption for continuous, fixed-geometry comparison; and the three-part falloff-orientation-eigenvalue discrimination test separating a genuine tidal signature from Earth's own centrifugal contribution.

The kill criterion named in this project's registration — that residuals must localize an injected nonintegrable shift, and that an inferred potential must agree with independent gravimetry — is stated precisely enough to fail by (Sections 3 and 4 give the exact quantity a fault-localization test would apply to; Section 7 gives the exact sensitivity an independent-gravimetry comparison would need to beat), but neither half has been run against real or even synthetic network data, and Section 10 names both as open rather than assumed favorable.

What survives, narrowly, is a structural fact rather than a general theorem: a clock network's one edge datum, the log frequency ratio, does not have one intrinsic relationship to "gravity." It has three cleanly separated relationships — to loop consistency, to potential slope, and to curvature proper — and this draft has shown, by proof where a proof was available and by exact evaluation where a proof was not the right tool, exactly which order of derivative each one accesses and which it does not. A future account that reports a single number as "the network detected curvature" without first stating which of these three quantities produced it would be quietly collapsing a distinction this draft was built, and checked wherever an exact check was available, to keep apart.

A An Independent Route via Geodesic Deviation

Section 5 derives $R^i{}_{0j0} = U^2(\phi_{,ij} + \phi_{,i}\phi_{,j})$ from the Christoffel symbols directly. This appendix rederives the same physical content by the geodesic deviation equation instead, giving an independent check that Eq. (15) is the standard tidal tensor of two nearby free-falling test masses, not an artifact of the specific coordinate computation.

For a geodesic congruence with tangent u^μ and connecting vector ξ^μ, the geodesic deviation equation is

$$\frac{D^2\xi^\alpha}{d\tau^2} = -R^\alpha{}_{\beta\gamma\delta}\, u^\beta \xi^\gamma u^\delta. \tag{22}$$

Consider two test masses momentarily at rest relative to the static observers of Section 5's metric $ds^2 = -U(\mathbf{x})^2(dx^0)^2 + \delta_{ij}dx^i dx^j$, separated by a small, purely spatial ξ^i at the instant they are released to free fall. At that instant $u^\mu = (u^0, 0, 0, 0)$ with $u^0 = 1/U$ (from $d\tau = U\, dt$ and $dx^0 = c\, dt$, so $u^0 = dx^0/d\tau = c/U$ in these units), so in Eq. (22) only $\beta = \delta = 0$ survives:

$$\frac{D^2\xi^i}{d\tau^2} = -R^i{}_{0j0}\,(u^0)^2\,\xi^j = -c^2\,R^i{}_{0j0}\,\xi^j \tag{23}$$

at a reference point where $U = 1$ is chosen for convenience (any point can be so normalized without loss of generality, since $\mathcal{T}_{ij}$ of Eq. (16) is already U^2-independent up to the negligible factor noted in Proposition 6.1). Substituting Eq. (15),

$$\frac{D^2\xi^i}{d\tau^2} = -c^2\big(\phi_{,ij} + \phi_{,i}\phi_{,j}\big)\xi^j = -\mathcal{T}_{ij}\,\xi^j, \tag{24}$$

exactly the standard relative tidal acceleration between two freely falling test masses, reducing at weak field to the familiar Newtonian $\ddot{\xi}^i = -\Phi_{,ij}\xi^j$ used throughout gravity-gradient geodesy [16]. This is the same object as Eq. (16), arrived at without ever writing a Christoffel symbol, confirming that $\mathcal{T}_{ij}$ is the physically standard tidal tensor rather than a construction specific to the lapse-function computation of Section 5.

For the Rindler metric, two test masses released from rest relative to the accelerating elevator immediately become inertial in the underlying flat spacetime and travel on straight lines; their proper separation, measured in the momentarily comoving inertial frame at release, does not change to any order. Eq. (24) therefore requires $\mathcal{T}_{ij} = 0$ identically for the elevator by the same physical reasoning Proposition 5.1 reaches algebraically — a genuinely independent confirmation, from the definition of curvature as relative free-fall acceleration rather than from the Christoffel-symbol computation, that a uniformly accelerated frame manufactures a slope in ϕ without manufacturing any relative acceleration between free-falling neighbors at all.

B A Ledger of Every Exact Check Performed

`check_clock_network_curvature.py` performs no fault-injection Monte Carlo study, no cross-validation against gravimetry, and no entanglement-covariance test. In its default `-self-check` mode it performs the following, each described in the sections above:

1. antisymmetry of the edge datum, $y_{ij} = -y_{ji}$, confirmed to floating-point precision on an explicit pair of frequencies (Section 1);

2. exactness of the weighted least-squares residual (Proposition 3.1) on the 3-node triangle and 4-node kite graphs for a randomly drawn node potential, together with the raw cycle-sum identity of Eq. (7), and confirmation that the fitted residual is identical (to machine precision) regardless of which of the graph's nodes is chosen as the gauge-fixing ground (Section 2, Proposition 6.1);

3. the discrete-holonomy identity of Corollary 3.1: injecting a fixed, non-conservative offset on one edge of the kite graph, the unaffected fundamental cycle's raw sum stays exactly zero, the affected cycle's raw sum equals the injected offset exactly with no fit performed, and the fitted residual's cycle sum over the affected loop equals the same value under four different grounding choices;

4. that a connected spanning tree carrying uniformly random, mutually inconsistent edge data is fit with exactly zero residual (Corollary 3.2);

5. a symbolic dimensional-analysis check (tracking length and time exponents with no third-party units library) confirming the edge datum and the weak-field redshift formula are dimensionless, that the closed-loop Sagnac holonomy of Eq. (9) is a time rather than a dimensionless fractional frequency, and that $\mathcal{T}_{ij}$ carries units of inverse time squared (Sections 4, 6);

6. exact evaluation of $\phi'' + (\phi')^2$ from the closed-form Rindler lapse at five heights and accelerations, together with an independent central finite-difference evaluation of the same combination directly from $\phi(Z)$, both vanishing to the expected floating-point and $O(h^2)$ tolerances respectively (Proposition 5.1);

7. finite differencing of $\Phi(x, y, z) = -GM_\oplus/\sqrt{x^2 + y^2 + z^2}$ at Earth's surface, confirming the Hessian is trace-free to six significant figures and reproduces the closed-form $-2:1:1$ pattern of Eq. (18) to the same precision (Section 7);

8. closed-form arithmetic evaluation of the radial tidal magnitude, the required clock-separation heights at $\sigma_y = 10^{-18}$ and $\sigma_y = 7.6 \times 10^{-21}$, the centrifugal-to-tidal ratio and percentage of Section 8, the closed-loop Sagnac holonomy of Eq. (10), and the Chou et al. (2010) 33 cm redshift, each reproducing the number quoted in the running text.

The saved output, `scaffold_validation_output.txt`, is the program's real captured standard output from a `-self-check` run; `scientific_result` and `physics_fault_injection_simulation_run` are both recorded as `false`, and no field reports a fault-localization confidence, a gravimetry comparison, or a covariance-improvement bound as a physical measurement. The program's `-simulate` option exits with status 2 and prints the seven-item list of unimplemented prerequisites named in Section 10; it never proceeds to produce a number. A later implementation must not reuse the name `check_clock_network_curvature.py` while silently relaxing these checks; it must add the fault-injection, gravimetry-comparison, covariance-bound, non-spherical-Earth, and gravitomagnetic-estimate tests named in Section 10, adopt a versioned output schema, and replace `requirements.lock` with an exact, platform-tested dependency lock for whichever network-simulation stack that study requires.

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